we have demonstrated the usefulness of the type of dampers investigated in Ref. 3 for vibration control of periodic structures. Such dampers may be tuned to suit the spectral shape of a noise environment.

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A Simplified Model for Aircraft Steering Dynamics

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THIS Note describes a simplified dynamical model of an aircraft that is useful for combat simulations involving aircraft steering. Both translational and rotational equations are developed. The former are obtained by expanding the equations of motion along aircraft flight path axes. The latter, however, are obtained only from geometrical considerations and transfer functions for the lift magnitude and lift bank-angle. In this sense, the equations are simplified.

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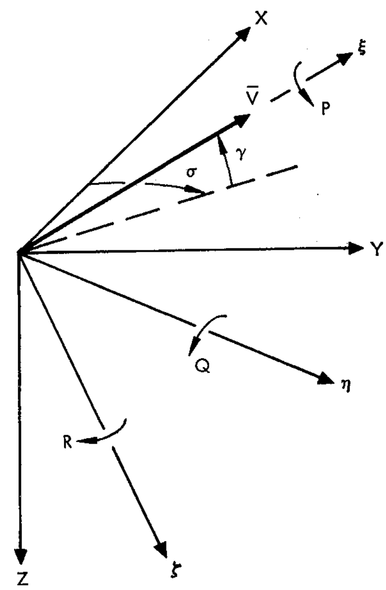

Fig. 1 Aircraft velocity vector.

Analysis

Flat Earth assumptions are made so that a local vertical, Earth-fixed reference frame is considered an inertial frame. The frame axes are X, Y, Z, where XY are in the local horizontal plane and Z is down along local vertical. Aircraft velocity vector $\mathbf{V}$ is expressed in terms of its magnitude V, and azimuth and elevation flight-path angles σ, γ respectively (Fig. 1). Angle σ is measured from the X axis to the projection of $\mathbf{V}$ in the XY plane, γ is the flight-path angle of $\mathbf{V}$ above the XY plane. Introduce axes ξ, η, ζ where ξ lies along $\mathbf{V}$, η is perpendicular to ξ and lies in the XY plane, and ζ completes a right-handed system. (If $\mathbf{V}$ lies along $\pm Z$, then η, ζ are ill-defined. This special case is examined in the Appendix).

The forces which act on the aircraft are lift L, drag D, thrust T and gravity W (Fig. 2). L is always perpendicular to $\mathbf{V}$ and so lies in the $\eta\zeta$ plane at some angle ρ to the $-\zeta$ axis. Drag

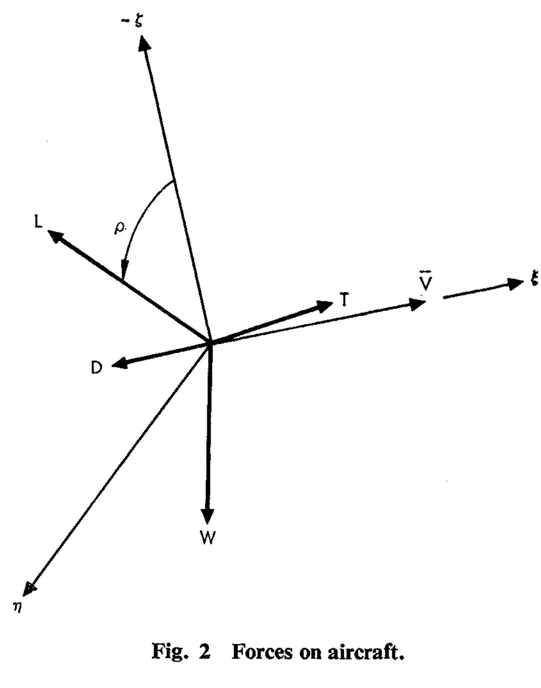

Fig. 2 Forces on aircraft.

lies along $-\xi$. Thrust lies nearly along ξ and, in general, has components T_ξ, T_η, T_ζ. The components of gravity along ξ, η, ζ are proportional to $(-\sin\gamma, 0, \cos\gamma)$.

Let a_ξ, a_η, a_ζ be the components along ξ, η, ζ of the aircraft acceleration vector **a**. The law of motion expanded along ξ, η, ζ yields

$$\begin{aligned} a_\xi &= \dot{V} = -a_D + a_{T\xi} - g\sin\gamma \\ a_\eta &= V\dot{\sigma}\cos\gamma = a_L\sin\rho + a_{T\eta} \\ a_\zeta &= -V\dot{\gamma} = -a_L\cos\rho + a_{T\zeta} + g\cos\gamma \end{aligned} \tag{1}$$

where

$a_L = (L/m)$, lift acceleration; $a_D = (D/m)$, drag acceleration

$a_T = (T/m)$, thrust acceleration; $a_{T\xi} \simeq a_T$; $a_{T\eta}, a_{T\zeta} \ll a_T$

The first of Eqs. (1) can be considered as an equation for aircraft speed V for some specified thrust acceleration a_T. This equation would be used in a simulation that employs speed variations as part of the maneuvers. For steering purposes, however, speed control is secondary to directional control so for simplicity the assumption is made that V is constant at some prescribed initial value. This eliminates the first of Eqs. (1).

The second and third of Eqs. (1) are the steering equations. The lift acceleration a_L and bank angle ρ are the control variables; they are generated in accordance with the demands of some steering law and the aerodynamic capabilities of the aircraft. The small thrust components $a_{T\eta}$, $a_{T\zeta}$ are considered as perturbations, and can be approximated as constants or simple functions of time. Rewriting these equations

$$\begin{aligned} \dot{\sigma} &= (a_L\sin\rho + a_{T\eta})/V\cos\gamma \\ \dot{\gamma} &= (a_L\cos\rho - a_{T\zeta} - g\cos\gamma)/V \end{aligned} \tag{2}$$

which, upon integration, determines the flight-path angles σ, γ. For $\gamma \to \pm 90°$, it appears that $\dot{\sigma} \to \infty$, but actually one has a 0/0 condition which can be resolved to produce a unique limit. This is discussed in the Appendix.

With σ, γ known the Cartesian components of aircraft velocity are

$$\dot{X} = V\cos\gamma\cos\sigma;\ \dot{Y} = V\cos\gamma\sin\sigma;\ \dot{Z} = -V\sin\gamma \tag{3}$$

which can be integrated to determine aircraft position. Thus, the translational motion of the aircraft is completely determined by the time-history of a_L and ρ, and initial conditions on V (assumed constant), σ, γ, and the position components.

The rotational motion of the aircraft is determined in the following way. The key assumption is zero sideslip, which means that the velocity and the lift always lie in the aircraft symmetry plane. Further, **V** makes an angle of attack, α, with respect to the aircraft roll axis. Designate the aircraft body axes as x, y, z where in the usual way, $x =$ roll, $y =$ pitch, $z =$ yaw. The transformation from ξ, η, ζ to x, y, z is then

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = [2, \alpha][1, \rho]\begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} \tag{4}$$

where $[2, \alpha]$, $[1, \rho]$ are transformation matrices. The generic notation is

$$[1, \theta_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\theta_1 & S\theta_1 \\ 0 & -S\theta_1 & C\theta_1 \end{bmatrix},\quad [2, \theta_2] = \begin{bmatrix} C\theta_2 & 0 & -S\theta_2 \\ 0 & 1 & 0 \\ S\theta_2 & 0 & C\theta_2 \end{bmatrix}$$

$$[3, \theta_3] = \begin{bmatrix} C\theta_3 & S\theta_3 & 0 \\ -S\theta_3 & C\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5}$$

where θ_1, θ_2, θ_3 are arbitrary angles and $C\theta_1 \equiv \cos\theta_1$, $S\theta_1 \equiv \sin\theta_1$, etc.

By virtue of the transformation in Eq. (4), it follows that the angular velocity of the aircraft, ω, equals the angular velocity of the $\xi\eta\zeta$ system, Ω, plus the vectorial addition of $\dot{\rho} \equiv \nu$ and $\dot{\alpha}$. Designate unit vectors along the body axes as **i**, **j**, **k** and the ω components as p, q, r. Then

$$\omega = \mathbf{i}p + \mathbf{j}q + \mathbf{k}r = \mathbf{\Omega} + \mathbf{i}_\xi\dot{\rho} + \mathbf{j}\dot{\alpha} \tag{6}$$

where $\mathbf{i}_\xi$ is the unit vector along axis ξ. Taking components in Eq. (6) along **i**, **j**, **k** and using transformation (4) yields

$$\begin{aligned} p &= (P + \nu)\cos\alpha + Q\sin\rho\sin\alpha - R\cos\rho\sin\alpha \\ q &= Q\cos\rho + R\sin\rho + \dot{\alpha} \\ r &= (P + \nu)\sin\alpha - Q\sin\rho\cos\alpha + R\cos\rho\cos\alpha \end{aligned} \tag{7}$$

where P, Q, R are the components of $\mathbf{\Omega}$ along the ξ, η, ζ axes and are given by Fig. 1

$$P = -\dot{\sigma}\sin\gamma;\ Q = \dot{\gamma};\ R = \dot{\sigma}\cos\gamma \tag{8}$$

The expression for Q is obvious; those for P and R follow from the fact that their resultant is $\dot{\sigma}$ which lies along the Z axis.

The remainder of the rotational kinematics depends on determining ρ, $\nu \equiv \dot{\rho}$, α, and $\dot{\alpha}$. From Fig. 2 it is clear that the aircraft velocity vector **V** is steered by controlling a_L and ρ. One can now imagine some steering law such as proportional navigation or lead pursuit which determines a desired direction for the aircraft velocity which, in turn, implies values of commanded lift acceleration $\hat{a}_L$ and commanded bank angle $\hat{\rho}$. Achieved acceleration a_L and achieved lift bank angle ρ can then be related to their commanded values by means of transfer functions, where the parameters would be aerodynamically dependent. Thus $(a_L/\hat{a}_L)$, a longitudinal aerodynamic transfer function, would probably be chosen to exhibit the short period motion, $(\rho/\hat{\rho})$ a lateral transfer function, could be chosen to reflect the uncoupled roll mode.

With a_L known, angle of attack α can be determined from the equation

$$C_L q_a S = L = m a_L$$

or

$$C_L(\alpha) = (m a_L/q_a S) \tag{9a}$$

where $C_L =$ lift coefficient, $m =$ aircraft mass, $q_a =$ dynamic pressure and $S =$ reference area. If the usual assumption that C_L is linear with α can be invoked, Eq. (9a) can easily be solved for α. To obtain $\dot{\alpha}$, differentiate Eq. (9a) with respect to time.

$$\dot{\alpha}(dC_L/d\alpha) = (m/S)(d/dt)(a_L/q_a) \simeq (m/q_a S)\dot{a}_L \tag{9b}$$

since q_a is relatively slowly changing. In Eq. (9b), $\dot{a}_L$ is obtained by differentiating the $a_L(t)$ response. The ρ, ν responses are obtained directly from the $(\rho/\hat{\rho})$ transfer function.

The body rates p, q, r given by Eq. (7) are now completely determined and the aircraft attitude angles ψ, θ, ϕ can then be found by integrating

$$\begin{aligned} \dot{\psi} &= (1/\cos\theta)(q\sin\phi + r\cos\phi) \\ \dot{\theta} &= q\cos\phi - r\sin\phi;\ \dot{\phi} = p + \dot{\psi}\sin\theta \end{aligned} \tag{10}$$

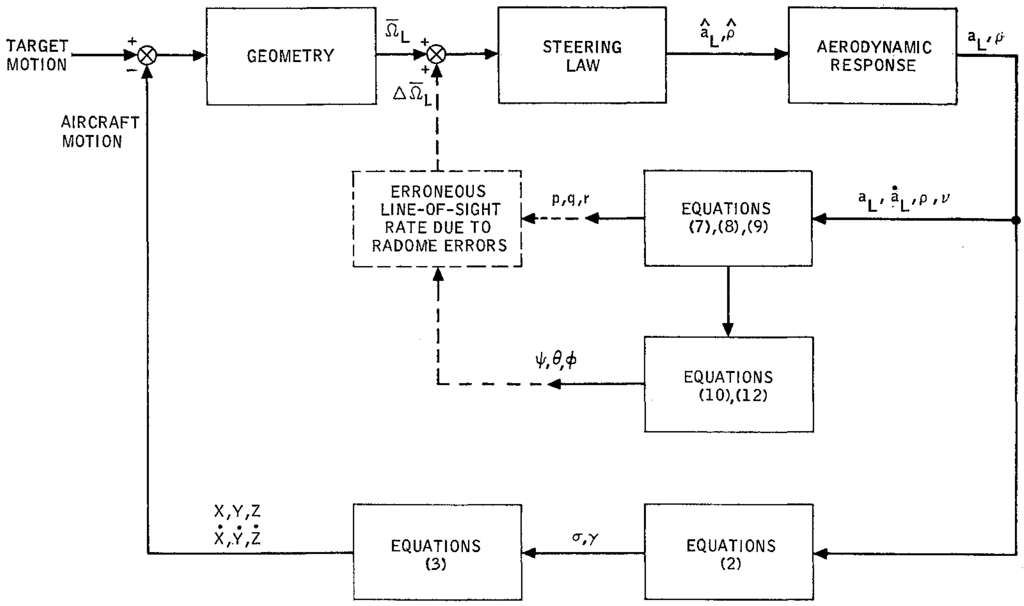

Fig. 3 Block diagram of steering simulation.

Initial conditions on ψ, θ, ϕ are determined from those on σ, γ, ρ and α as a result of the matrix equivalence [see notation of Eq. (5)]

$$[2, \alpha][1, \rho][2, \gamma][3, \sigma] = [1, \phi][2, \theta][3, \psi] \tag{11}$$

Equation (11) states that the transformation from XYZ to aircraft body axes xyz can be acheived in two ways: 1) through the conventional Euler angles or 2) through the flight-path angles σ, γ, the lift bank-angle ρ, and the angle of attack α. Solving Eqs. (11) for ψ_o, θ_o, ϕ_o and using the small-angle properties of α

$$\begin{aligned} \psi_o &= \sigma_o + \alpha_o \sin \rho_o / \cos \gamma_o; \; \theta_o = \gamma_o + \alpha_o \cos \rho_o \\ \phi_o &= \rho_o + \alpha_o \sin \rho_o \tan \gamma_o \end{aligned} \tag{12}$$

Equations (12) are used only for initial values; for $t > 0$, ψ, θ, ϕ are obtained by integrating Eqs. (10).

Figure 3 shows how all the relations developed here are linked together to form an over all block diagram for simulating the steering problem. The target motion is usually a prescribed function of time although it can be generated from some steering law. The geometry calculation computes quantities such as range from interceptor to target, range-rate, and line-of-sight rate $\mathbf{\Omega}_L$.

As presently structured, the aircraft rotational motion, $p, q, r, \psi, \theta, \phi$ is shown as an auxiliary calculation which has no effect on the stability of the steering loop. However, there are refinements to the steering problem, whose inclusion requires the aircraft rotational motion and which would then effect steering stability. For example, if the aircraft carries a tracking radar the measured line-of-sight rate of the target as seen by the radar will be corrupted by radome refraction errors and their slopes.[1] Evaluation of this effect requires explicit histories of aircraft rates p, q, r, attitude ψ, θ, ϕ, and measured values of the radome errors and slopes. This calculation generates an error, $\Delta\mathbf{\Omega}_L$, in the line-of-sight rate which combines with the true $\mathbf{\Omega}_L$ to produce the total rate sensed by the tracking radar. Figure 3 shows that inclusion of the radome effects produces an inner loop which can have a pronounced effect on steering stability.

Conclusions

A set of dynamical equations useful for aircraft combat simulations has been developed. The translational equations are particularly efficient for steering problems since there are only two variables σ, γ which determine the direction of the interceptor aircraft velocity. Rotational dynamics are omitted and replaced by three conditions: 1) zero sideslip, 2) a transfer function relating achieved acceleration to commanded acceleration, 3) a transfer function relating achieved lift bank-angle to commanded bank-angle. The former transfer function is representative of aircraft longitudinal dynamics, the latter, of lateral dynamics. Finally, it has been shown how to incorporate these relations into a steering simulation. It is believed that this particular set of dynamical equations is relatively simple but still allows for inclusion of all significant aircraft dynamics.

Appendix

The ξ, η, ζ system is well defined so long as $\gamma \neq 90°$ (or $-90°$). However, at $\gamma = 90°$ an ambiguity exists since the $\mathbf{i}_\xi$ g plane degenerates to a line. As in all problems of this type, the true situation is unknown; the only plausible procedure is to seek limiting values and then define all variables to have these limiting values.

So long as $\gamma \neq 90°$ the angle between η and the X axis is $(90 + \sigma)°$ (Fig. 1). Since this is obviously independent of γ, this must still be the angle when $\gamma = 90°$. Thus for $\gamma = \pm 90°$, ξ points along the $\mp Z$ axis, η lies in the XY plane at $(90 + \sigma)°$ to the X axis, where σ is the limiting value, and ζ is perpendicular to ξ and η in the sense of the right-hand rule; thus ζ lies in the XY plane.

The equation $\dot{\sigma} = a_\eta/(V \cos \gamma)$ shows that as $\gamma \to 90°$, $\dot{\sigma} \to \infty$. However, since there is no physical reason for $\dot{\sigma}$ to be infinite, the kinematics must be self-adjusting such that $a_\eta \to 0$ as $\gamma \to 90°$. This is actually so, by the following argument.

As $\mathbf{V}$ sweeps across the pole, the acceleration of $\mathbf{V}$ must be entirely in the $\xi\zeta$ plane, i.e., $a_\eta = 0$, for if $a_\eta \neq 0$, then $\mathbf{V}$ would not go directly over the top but would "wobble" around the pole. With $a_\eta = 0$, and $\gamma \to 90°$,

$$\dot{\sigma} = a_\eta / V \cos \gamma \to 0/0 = ?$$

Thus $\dot{\sigma}$ is indeterminate. To resolve the indeterminacy, apply L'Hospitals rule. Using $(d/d\gamma) = (\dot{\gamma})^{-1}(d/dt)$

$$(\sigma)_{\gamma \to 90°} \to -\dot{a}_\eta / V \dot{\gamma}$$

Thus when γ is within some ϵ of $\pm 90°$, Eqs. (2) should be replaced by

$$\dot{\gamma} = (a_L \cos \rho - a_{T\zeta} - g \cos \gamma)/V$$
$$\dot{\sigma} = -(d/dt)(a_L \sin \rho + a_{T\eta})/V\dot{\gamma} \quad (13)$$

It is conventional in this special case to allow a jump in σ of 180° and maintain $\gamma < 90°$. However, if $|\gamma| > 90°$ is permitted, then using Eqs. (13), $\dot{\sigma}$ can be integrated smoothly through the $\gamma = 90°$ region without any resulting jump in σ.

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